

*Rapid Note***Self organization of interacting polya urns**M. Marsili^{1,a} and A. Valleriani²¹ International School for Advanced Studies (SISSA) and INFN unit, V. Beirut 2-4, 34014, Trieste, Italy² Max-Planck-Institut für Physik Komplexer Systeme, Nöthnitzer Str. 38, 01187 Dresden, Germany

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Abstract. We introduce a simple model which shows non-trivial self organized critical properties. The model describes a system of interacting units, modelled by Polya urns, subject to perturbations and which occasionally break down. Three equivalent formulations – stochastic, quenched and deterministic – are shown to reproduce the same dynamics. Among the novel features of the model are a non-homogeneous stationary state, the presence of a non-stationary critical phase and non-trivial exponents even in mean field. We discuss simple interpretations in term of biological evolution and earthquake dynamics and we report on extensive numerical simulations in dimensions $d = 1, 2$ as well as in the random neighbors limit.

PACS. 64.60.Ht Dynamic critical phenomena – 64.60.Lx Self-organized criticality; avalanche effect

Our understanding of Self Organized Criticality (SOC) [1], as a general framework for the emergence of scale-free behavior in Nature, has greatly benefitted from the introduction of simple models. Even though models such as the sandpile [1] and the Bak-Sneppen [2] are too simple to capture the complexity of natural phenomena such as earthquakes [3] and biological evolution [4,5], they have, nonetheless, identified some basic mechanisms leading to SOC. These systems have been a starting point both for the development of more complex and realistic models of natural phenomena [6,7], and for analytical approaches [8–11], which have led us to a much deeper understanding of SOC. Indeed, we can now identify some basic “routes to Self Organized Criticality” such as those based on sandpile [1], extremal dynamics [10,12], memory [13] and network [14] models.

In this Rapid Communication we propose a qualitatively different “route to SOC” based on a very simple model of interacting Polya urns. Its qualitative differences with respect to other SOC models are that it is characterized by a non-homogeneous stationary state and by non trivial exponents even in the mean field case. Furthermore, we shall show numerical evidence for the occurrence of a *non-stationary self organized critical state*. Moreover, the model can be formulated in three different but equivalent ways. This fact, on one hand allows us to use a wide variety of tools to investigate its critical properties, and on the other it bridges different descriptions of the same process. All these features can well be relevant in the description of natural phenomena. The model indeed provides a general

framework for the emergence of SOC which, as we shall discuss, can be applied both to coevolution and to large scale earthquakes dynamics. Note indeed that the patterns of earthquakes activity are highly non-homogeneous and that such a system is, in principle, non-stationary. The same applies to our ecosystem, which is in a non-stationary state where ever fitter species replace less fit ones.

We consider a system of interacting Polya urns arranged on a d -dimensional lattice. A Polya urn is a simple model to study *e.g.* the occurrence of accidents [16]. Each urn contains initially b black balls and 1 white one. As in sandpile models, at each time step we randomly select a site and attempt to add a “grain of sand”, *i.e.* a white ball, to the corresponding urn. A ball is drawn from the selected urn: if the ball is white the attempt is successful and a new white ball is added to the urn. If it is black a “fatal accident” occurs: the urn becomes unstable and it “topples”. The toppling mechanism is as follows: 1) the urn is reset to 1 white ball and b black ones and 2) for each white ball of the original urn a similar attempt to add a white ball is made on a randomly chosen nearest neighbor urn. In this way, white balls released by an unstable urn can provoke some “fatal accident” in nearby urns (addition of white ball to already unstable urns leaves them unstable but it increases the number of white balls in it). The process stops when all balls are redistributed provoking no further toppling. A new attempt to add a ball to a randomly chosen urn is made, at the next time-step, and the process goes on. Dissipation of balls at the boundary, as in the sandpile [1], can also be considered to allow the

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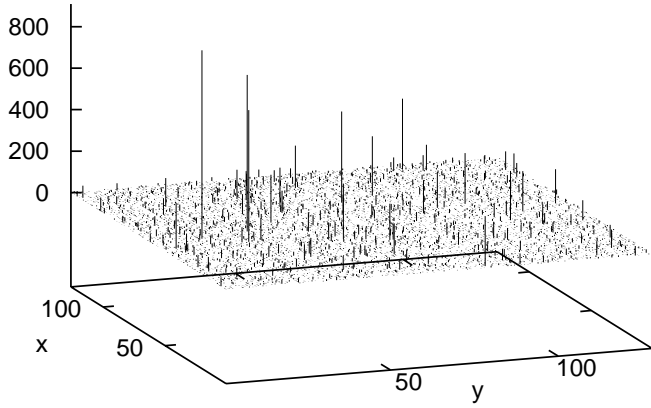


Fig. 1. Snapshot of a section of the $d = 2$ system of size $L = 128$ for $b = 2.50$ and $\lambda = 10^{-2}$.

system to relax to a stationary state. Actually, in order to keep the same definition of the model both in finite dimensions and in the random neighbor version, we consider here “bulk dissipation” modifying step 2) into: 2’) with probability λ all white balls disappear, otherwise 2) applies.

Let e_i be the number of subsequent additions of white balls in urn i , since the last draw of a black one. Then urn i contains b black balls and $e_i + 1$ white ones. The toppling probability, *i.e.* the probability to draw a black ball from urn i , is then

$$f_i = \frac{b}{e_i + b + 1}. \quad (1)$$

In general, the probability that urn i topples, if it receives q white balls from neighbors, is

$$w_i(q) = 1 - \frac{(e_i + q)! \Gamma(e_i + b + 1)}{e_i! \Gamma(e_i + q + b + 1)}. \quad (2)$$

The Γ -function is introduced here in order to generalize our discussion to any real positive b .

The system spontaneously evolves to a critical state which is generally characterized by a non uniform distribution of the variables e_i . A snapshot of the system for $b = 2.5$ is shown in Figure 1. It is clearly seen that very stable urns $e_i \gg 1$ coexist with less stable ones.

We say that the initial perturbation causes an *avalanche* of size s and volume v , where s is the number of toppling events occurring before the system returns stable and v is the number of distinct sites involved in the avalanche. Operationally all urns which become unstable after the first event topple simultaneously. The urns, which as a result of this first wave of topplings become unstable, topple simultaneously in a second wave, and so on. An avalanche is then also characterized by the number w of such waves occurring before the system returns stable. Finally, one can also measure the total number E of white balls involved in the avalanche event. Of course, if the initial urn resists absorbing an extra white ball, $s = v = w = E = 0$. The distribution of avalanche sizes in the critical state

$$P(s) \sim s^{-\tau} \quad (3)$$

has a power law behavior. In this state the volume, waves and the number E of the avalanche are related to its size by power laws

$$v \sim s^{d/z}, \quad w \sim s^\omega, \quad E \sim s^\gamma. \quad (4)$$

The four exponents τ , z , ω and γ define the critical state of the model. They are given in Table 1 for several values of $b > 1$ in $d = 1$, $d = 2$ and in the random neighbors version (K “neighbors” are randomly chosen each time among all sites) respectively¹. For $b \leq 1$ the system becomes non-stationary: the number of white balls in the system increases linearly in time. In spite of this, we have found that avalanches have still a well defined distribution. We have analyzed in particular the border-line case $b = 1$: for the random neighbor model, we have found that in a range of times (from 10^5 to 2×10^8), for various system sizes (up to $n = 2^{14}$ urns) and different dissipation rates ($\lambda = 10^{-2}$ and $\lambda = 10^{-3}$), the distribution of avalanche sizes follows equation (3) on more than three decades, with an exponent which is, for all these cases, always in the range $\tau \in [1.95, 2.05]$. In $d = 2$, for sizes up to $n = 2^{14}$ and in a range of times from 5×10^7 to 3×10^8 , we have similarly found $\tau \in [2.03, 2.11]$. In $d = 1$, extensive simulations over lattice lengths of 256 and 512 sites and on a range of times from 10^9 to 6×10^9 , we have found $\tau \in [2.10, 2.21]$.

In an ecosystem species are probed by changes in the environment in a fashion which goes under the name of co-evolution. Namely, the environment, which is constituted by the interaction among all living organisms, is implicitly modified by each single being and determines its fate as well². From the point of view of any single species, such perturbations impose a random selective pressure which eventually leads to a change in the constitution of the species or to its extinction. A particular character, developed by a random mutation in a subspecies, can be “selected” by evolution if it becomes essential for the survival of the whole species in a changed environment [4]. In this process, which is driven by chance, species become more and more complex. A more complex species has also a wider variability in subspecies which, in this process, will make it more resistant. In much the same way, Polya urns in the model “evolve” increasing their “complexity” and their robustness to external perturbations. The model also assumes that the extinction of well adapted “species” (*i.e.* urns with $e_i \gg 1$) produces a larger perturbation than that caused by poorly fit species.

In this toy ecosystem, each species has exactly the same chances of any other species to survive, when it appears ($e_i = 0$). There is no *a priori* genetic characteristic,

¹ The parameter λ , which introduces bulk dissipation, sets the upper cutoff of the critical states. We found that, in $d = 1, 2$, dissipation at the boundaries, as in the sandpile model, leads to similar results. In the random neighbors version, we have found that the exponents do not depend on the number K of neighbors chosen.

² Just as an example, one could think at the selection pressure produced by the presence of oxygen in the air, which has been initially produced by some zoo-phyte.

Table 1. The exponents for different values of b for $d = 1$ and sizes up to $L = 512$, $d = 2$ and $L = 128$, and for the random neighbor model with $n = 16384$ sites and $K = 2$.

b	$d = 1$				$d = 2$				$d = \infty$	
	τ	z	ω	γ	τ	z	ω	γ	τ	ω
1.40	1.80(2)	2.45(1)	0.61(1)	1.44(1)	1.83(1)	3.17(2)	0.44(1)	1.248(3)	1.885(7)	0.34(3)
1.60	1.69(2)	2.35(1)	0.60(1)	1.36(1)	1.74(1)	3.10(2)	0.45(1)	1.183(3)	1.80(1)	0.38(2)
1.80	1.61(2)	2.29(1)	0.60(1)	1.31(1)	1.68(1)	3.03(2)	0.45(1)	1.134(1)	1.68(1)	0.40(2)
2.00	1.55(2)	2.24(1)	0.60(1)	1.27(1)	1.63(1)	2.97(2)	0.46(1)	1.103(1)	1.667(7)	0.42(1)
2.30	1.47(2)	2.18(1)	0.59(1)	1.23(1)	1.57(1)	2.90(2)	0.48(1)	1.069(1)	1.642(9)	0.44(1)
2.50	1.43(2)	2.15(1)	0.59(1)	1.21(1)	1.55(1)	2.84(2)	0.49(1)	1.056(1)	1.624(8)	0.45(1)
2.70	1.39(2)	2.13(1)	0.59(1)	1.19(1)	1.54(1)	2.81(2)	0.50(1)	1.045(2)	1.609(8)	0.47(1)
3.00	1.36(2)	2.11(1)	0.59(1)	1.17(1)	1.52(1)	2.76(2)	0.51(1)	1.034(3)	1.595(7)	0.47(1)

such as fitness, which guarantees the survival of a species. Its survival will rather depend on its “ability” to adapt constantly, *via* random mutations, to the changing environment. This perspective, also suggests that an operational measure of fitness (or resistance) of a species, is possible using equation (1): high e_i means high fitness. Note that this differs from the concept of fitness introduced in most SOC models of coevolution [2, 7, 14]. In these, fitness is related to reproduction rates, whereas in our simplified picture of coevolution, fitness emerges as a measure of the resistance against extinction of a species.

We show now that this different notion of fitness, when introduced as an *intrinsic* property of each species, leads to exactly the same coevolutionary process. To be more precise, let us assume that the probability f_i of extinction of species i under a perturbation, is no more given by equation (1), but it is rather fixed for each species. In particular, this *intrinsic* property is drawn randomly for each species from a distribution $\rho(f)$. Accordingly we also replace equation (2) by $w_i(q) = 1 - (1 - f_i)^q$. Species are probed by random perturbations (addition of white balls) and, as before, the extinction of species i perturbs “neighboring” species in the interaction web *via* the same toppling mechanism. When a species disappears, its “niche” (site) is immediately occupied by a new species, with a new randomly drawn fitness value f'_i . Thus, for

$$\rho(f) = bf^{b-1} \quad (5)$$

we obtain exactly the same stochastic dynamics given by equations (1, 2). In order to show the equivalence, it is enough to show that equation (5) leads to the same rates $w_i(q)$ of equation (2). Consider one particular urn and let $e_{i,t}$ be the value of the corresponding variable after t drawings. The event $e_{i,t} = e$ occurs with a probability³ $P(e_{i,t} = e | e_{i,t-e} = 0, f_i) = (1 - f_i)^e$. Taking the average over $\rho(f)$, we find

$$P(e_{i,t} = e | e_{i,t-e} = 0) = \frac{\Gamma(b+1)e!}{\Gamma(e+b+1)}. \quad (6)$$

³ Here t is a “local” time on site i , which measures the number of perturbations on that site. Therefore the event $e_{i,t} = e$ implies $e_{i,t-e} = 0$. Accordingly we used the notation $P(e_{i,t} = e | e_{i,t-e} = 0, f_i)$ (note indeed that $P(e_{i,t} = 0 | e_{i,t} = 0, f_i) = 1$ for $e = 0$).

Using detailed balance, $P(e_{i,t+q} = e + q | e_{i,t-e} = 0) = w_i(q)P(e_{i,t} = e | e_{i,t-e} = 0)$, we easily recover equations (1, 2). The equivalence of the two models has been also tested in numerical simulations.

The initial definition of the model is completely stochastic, whereas in the formulation based on equation (5), f_i are fixed, quenched variables, which are renewed stochastically at each extinction event. The equivalence of the two descriptions is an example of a general mapping [12], recently developed to deal with extremal dynamics. Its application in the biological context are also discussed in reference [13].

There is a further interesting mapping, originally developed in the context of interface growth [17], which can be applied to the present model. As in the sandpile model [1], we define the toppling probability as

$$f_i = 0 \text{ if } e_i < h_i \text{ and } f_i = 1 \text{ if } e_i \geq h_i. \quad (7)$$

While in the sandpile the thresholds are fixed $h_i = 2d$, we introduce here a model where h_i are randomly drawn from a given distribution $\psi(h)$ after each toppling event on site i . The choice

$$\psi(h) = \frac{b\Gamma(b+1)\Gamma(h)}{\Gamma(h+b+1)} \quad (8)$$

reproduces a dynamics which is equivalent to the previous two formulations of the model. To prove this, it is enough to derive the statistics of the number $e_i = h_i - 1$ of perturbations that an urn with $f_i = f$ will overcome before toppling. Clearly $P(h_i = h | f_i = f) = f(1 - f)^{h-1}$. Taking the average over the distribution (5) of f leads indeed to equation (8).

This formulation is completely deterministic: it assumes that each urn appears with a prescribed “lifetime” measured in terms of perturbations. As soon as this lifetime is reached, the urn topples. This is the same threshold dynamics used in the sandpile model. Here however thresholds h_i are very broadly distributed (note that $\psi(h) \sim h^{-b-1}$), whereas in the sandpile $\psi(h) = \delta(h - 2d)$. The sandpile is a paradigm for seismic phenomena: each site represent a fault which is perturbed by the slow addition of stress energy. When the energy load e_i of a site exceeds the threshold h_i , the fault breaks down and all the energy is released to neighbor sites. Our model also

proposes a different description of the same phenomenon: each addition of stress energy has the same probability f_i to provoke an earthquake. When this occurs, it provokes the energy release and a local seismic rearrangement, *i.e.* $f_i \rightarrow f'_i$.

Using the quenched version of the model given by equation (5), it is easy to derive the effective distribution $\tilde{\rho}(f)$ of the f_i 's in the system at the stationary state. It is enough to consider detailed balance in an interval $f_i \in [f, f + df)$ under a single perturbation. The probability that, in one time step, one f_i leaves this interval is $f\tilde{\rho}(f)df$. This has to balance the number $\rho(f)df \int_0^1 df' f' \tilde{\rho}(f')$ of sites which enter this interval. This gives $\tilde{\rho}(f) = (b-1)f^{b-2}$. With equation (6) one can also derive the distribution of e_i in the system. Indeed, $P(e_{i,t} = e) = P(e_{i,t} = e | e_{i,t-e} = 0)P(e_{i,t-e} = 0)$ where $P(e_i = 0) = 1 - 1/b$ is derived imposing normalization. This leads to $P(e_i) \sim e_i^{-b}$.

As $b \rightarrow 1^+$, both the distributions $\tilde{\rho}(f)$ and $P(e_i)$ become unnormalizable and the probability of finding sites with $e_i = 0$ vanishes. Accordingly, numerical simulations show that, for $b \leq 1$, the system average of e_i increases linearly with time and the system never reaches a stationary state. The divergence of normalization of $\tilde{\rho}(f)$ at $f = 0$ occurs because less "fit" species are more rapidly replaced than more "fit" ones. For $b \leq 1$ the search for the "perfect" species $f_i = 0$ never stops. For $b > 1$, the probability that a perturbation causes a toppling is $P_{top} = \int_0^1 df f \tilde{\rho}(f) = \frac{b-1}{b}$, which also vanishes as $b \rightarrow 1$. This means that, for $b \leq 1$, avalanches occur more and more rarely as time goes on.

Let us discuss in more detail the random neighbor model. Numerical results are consistent with $z/D = 1$ and $\gamma = \max[1, 1/(b-1)]$, which is what one expects from the observation that each site is involved at most once in the same avalanche (the exponent $\gamma = 1/(b-1)$ for $1 < b < 2$ comes from the limit laws of Levy variables). On the other hand we see that the exponents τ and ω differ from their usual mean-field values $\tau = 3/2$ and $\omega = 1/2$, and that such values are eventually reached for $b \rightarrow \infty$. This deviation $\tau \neq 3/2$ can be understood as an effect of correlation. In order to show this, let us review the argument leading to $\tau = 3/2$. Consider an avalanche and let M_t be the number of unstable sites after t toppling events. M_t performs a random walk and the size $s = \min\{t : M_t = 0, t > 0\}$ of the avalanche is the first return time of M_t to 0. Therefore s has the same distribution of the first return times to the origin of a random walk $P(s) \sim s^{-3/2}$. Since the steps $|M_t - M_{t-1}|$ of the random walk are bounded by the coordination number K , the only possibility for a deviation from $\tau = 3/2$ is to have correlations. Correlations indeed arise because a toppling event may release many white balls and these are transported along the avalanche thus modifying the probability of toppling of successive sites. The theoretical calculation of the exponents for the random neighbor version is a challenging problem under current investigation.

In conclusion, we have introduced a very simple model which displays non-trivial self organized critical features.

The model was analyzed numerically and we also derived some analytic result. The main features of the model are non-homogeneous critical states, a critical non-stationary state in a region of the control parameter ($b \leq 1$) and non-trivial exponents even in the mean field limit. Furthermore, it allows for three different equivalent formulations, which allow one to better investigate and understand the nature of the critical state. As a closing remark, we notice that the non-stationary regime can be relevant both for the description of ecological systems and for earthquake dynamics. Both these systems are indeed not in a stationary state. Interestingly enough, the exponent τ is larger than 2 for $b < 1$. Both the Gutenberg-Richter law for earthquakes [3] and the extinction size distribution [5] also display avalanche exponents close to 2.

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